Published by Institute of Physics Publishing for SISSA/ISAS
Received: *June 6, 2002*
Accepted: *July 5, 2002*

# Some remarks on Feynman rules for non-commutative gauge theories based on groups $G \neq \mathrm{U}(N)$

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Abstract: We study for subgroups $G \subseteq \mathrm{U}(N)$ partial summations of the $\theta$-expanded perturbation theory. On diagrammatic level a summation procedure is established, which in the $\mathrm{U}(N)$ case delivers the full star-product induced rules. Thereby we uncover a cancellation mechanism between certain diagrams, which is crucial in the $\mathrm{U}(N)$ case, but set out of work for $G \subset \mathrm{U}(N)$. In addition, an explicit proof is given that for $G \subset \mathrm{U}(N)$, $G \neq \mathrm{U}(M)$, $M < N$ there is no partial summation of the $\theta$-expanded rules resulting in new Feynman rules using the $\mathrm{U}(N)$ star-product vertices and besides suitable modified propagators at most a *finite* number of additional building blocks. Finally, we show that certain $\mathrm{SO}(N)$ Feynman rules conjectured in the literature cannot be derived from the enveloping algebra approach.

Keywords: Non-Commutative Geometry, Gauge Symmetry.

## Contents





## 1. Introduction

In recent years there has been a lot of interest in gauge theories on non-commutative spaces and in particular in their relation to string theory, see [1, 2] for reviews.

The gauge theory based on the gauge transformation[1]

$$\delta_\Lambda A_\mu = \partial_\mu \Lambda - i A_\mu \star \Lambda + i \Lambda \star A_\mu \tag{1.1}$$

and the action

$$S[A] = -\frac{1}{2g^2} \int dx \ \ \mathrm{tr}(F_{\mu\nu} \star F^{\mu\nu}) \, , \tag{1.2}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star \, , \tag{1.3}$$

is classically consistent for the gauge group $\mathrm{U}(N)$ only [5]. This observation is based on the fact that the $\star$-commutator of two Lie algebra valued quantities is again in the Lie algebra only if the anticommutator of two generators is in the Lie algebra.

[1] We consider only the non-commutative version of Minkowski space $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ with constant $\theta^{\mu\nu}$ and use the Moyal $\star$-product formulation. To avoid problems with unitarity [3, 4] we restrict ourselves to the case of space non-commutativity.

On the string theory side one gets non-commutative gauge field theories in a certain infinite tension limit in the presence of a constant Neveu-Schwarz $B$-field [6]. Then, e.g. for the case of SO$(N)$, the problem reappears as an obstruction for the implementation of a $B$-field background for non-oriented strings. Attempts to overcome this obstruction have been made in [7, 8]. A discussion on the level of factorization properties of string tree amplitudes can be found in [9].

In spite of the obstruction preventing, for gauge groups $G$ other than U$(N)$, Lie algebra valued gauge fields $A_\mu$, one nevertheless can consistently construct non-commutative gauge field theories for all gauge groups within the enveloping algebra approach [10]. For subgroups of U$(N)$ this construction is equivalent to the following set up: First one expresses the non-commutative U$(N)$ gauge field and gauge transformation $\Lambda$ via the Seiberg-Witten map [6]

$$\begin{aligned} A_\mu &= a_\mu - \frac{1}{4}\,\theta^{\alpha\beta}\{a_\alpha, \partial_\beta a_\mu + f_{\beta\mu}\} + O(\theta^2)\,, \\ \Lambda &= \lambda + \frac{1}{4}\,\theta^{\alpha\beta}\{\partial_\alpha\lambda, a_\beta\} + O(\theta^2) \end{aligned} \tag{1.4}$$

in terms of a commutative U$(N)$ gauge field $a_\mu$ and gauge transformation $\lambda$, respectively. After that both $a_\mu$ and $\lambda$ are constrained to take values in the Lie algebra of the subgroup $G$ under discussion. In slightly different words the non-commutative version of the $G$ gauge theory is defined by (1.1)–(1.3) *and* (1.4) *as well as* the constraint $a,\ \lambda\ \in\ G$.

Inserting (1.4) into (1.2) and expanding the $\star$-product in powers of $\theta$ yields the following action for $a_\mu$

$$s[a] := S[A[a]] = -\frac{1}{2g^2}\int dx\,\mathrm{tr}(f_{\mu\nu}f^{\mu\nu}) + \cdots\,, \tag{1.5}$$

where the dots stand for an infinite series of terms containing higher derivatives and powers of $\theta$. After gauge fixing one immediately can read off Feynman rules for the field $a_\mu$. Besides the standard propagators and vertices for $a_\mu$ and the Faddeev-Popov ghosts one has an infinite set of additional vertices with an increasing number of legs, derivatives and powers of $\theta$. For our further discussion it is useful to stress that all these vertices are generated by $\theta$-expansion of the whole action, i.e. both the non-commutative kinetic and interaction term. In the following we call this kind of perturbation theory the *$\theta$-expanded perturbation theory* for the non-commutative $G$-gauge theory. It has been extensively studied e.g. in [11, 12, 13].

On the other side for the U$(N)$ case it is straightforward to get directly from (1.1)–(1.3) (after gauge fixing) Feynman rules in terms of $A_\mu$ and non-commutative Faddeev-Popov ghosts, see e.g. [14, 15]. The rules look very similar to that of standard (commutative) Yang-Mills theory. Modifications are due to the presence of anticommutators of Lie algebra generators, and the $\star$-product generates additional momentum dependent exponential factors in the vertices. These factors are responsible for the UV/IR effect [16]. This effect received a lot of attention in particular with respect to its stringy origin and its implications for the renormalization program. However, the UV/IR effect is not manifest in $\theta$-expanded perturbation theory for U$(N)$.



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For $G \neq \mathrm{U}(N)$, besides a conjecture for $\mathrm{SO}(N)$ in [17], Feynman rules in terms of the full non-commutative $A_\mu$ are not known. Our goal in this paper is to get information on these rules by studying some issues of partial summing the known $\theta$-expanded rules. Such rules would allow to study UV/IR mixing similar to the $\mathrm{U}(N)$ case.

One could also try to apply the full machinery of constrained quantization to the combined problem of gauge fixing *and* constraining to gauge fields and gauge transformations whose inverse Seiberg-Witten map is in $G$. However, as long as the SW map is available as a power expansion in $\theta$ only, there seems to be little hope to reach directly along this line Feynman rules with a finite number of building blocks.[2] The situation could be different for the $\mathrm{SU}(N)$ case, where an alternative constraint has been proposed in [18].

The paper is organized as follows. In section 2 we formulate our questions in precise technical terms. The original non-commutative interactions are kept as suppliers of at least part of the vertices of our wanted Feynman rules. The $\theta$-expanded perturbation theory is summed with respect to the vertices generated by the expansion of the kinetic term only.

In section 3 we study this program in parallel for both $\mathrm{U}(N)$ and $G \subset \mathrm{U}(N)$. Since the outcome for the $\mathrm{U}(N)$ case is a priori known, this case can serve as some check of the diagrammatic analysis. Indeed we will find a cancellation mechanism which guarantees the known result. On the other hand, this cancellation mechanism breaks down for subgroups $G \subset \mathrm{U}(N)$ which are not equal to some $\mathrm{U}(M)$, $M < N$. This already gives strong arguments for the nonexistence of Feynman rules based on the original non-commutative $\mathrm{U}(N)$ three and four-point vertices supplemented by suitable modified propagators and by at most a finite number of additional building blocks. These additional building blocks would be related to the connected Green functions of the gauge field and the ghosts obtained by summing the $\theta$-expansion of the kinetic term.

However, strictly speaking, by itself the absence of a mechanism doing the job in the $\mathrm{U}(N)$ case does not exclude some other mechanism enforcing the finiteness of the number of additional building blocks. Therefore, in section 4 we give an explicit proof that the partial summation of $\theta$-expanded perturbation theory, studied in the previous sections, yields non-vanishing Green functions with an arbitrary large number of external points.

Section 5 is devoted to a modification of the partial summation procedure designed to make contact with the $\mathrm{SO}(N)$ rules conjectured in ref. [17]. At least for $\mathrm{SO}(3)$ we will explicitly prove that these rules cannot be derived from $\theta$-expanded perturbation theory.

Some more technical considerations related to sections 4 and 5 can be found in appendices A and B.

## 2. The general framework

We start with non-commutative $\mathrm{U}(N)$ in Feynman gauge described in terms of the gauge field $A_\mu$ and Faddeev-Popov ghosts $C$ and $\bar{C}$. The Seiberg-Witten map for the ghost $C$ looks

[2]The $\theta$-expanded rules contain an infinite number of vertices. It seems to us of little use to replace these rules by another set of rules with an infinite number of vertices. Therefore, incited by the $\mathrm{U}(N)$ example, we are after rules with a finite number of building blocks.



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like that for the gauge transformation in (1.4) while the antighost is kept unchanged [11], i.e.

$$\begin{aligned} C &= c + \frac{1}{4}\,\theta^{\alpha\beta}\{\partial_\alpha c, a_\beta\} + O(\theta^2)\,, \\ \bar{C} &= \bar{c}\,. \end{aligned} \tag{2.1}$$

Then we separate

$$S[A,C,\bar{C}] = S_{\rm kin}[A,C,\bar{C}] + S_I[A,C,\bar{C}]\,, \tag{2.2}$$

with

$$S_{\rm kin}[A,C,\bar{C}] = -\frac{1}{g^2}\int dx\,{\rm tr}\,\partial_\mu A_\nu \partial^\mu A^\nu - \int dx\ \partial_\mu \bar{C}\partial^\mu C\,. \tag{2.3}$$

The generating functional for non-commutative $G$ Green functions is given by[3]

$$Z_G[J,\bar{\eta},\eta] = \int_{a,c,\bar{c}\in G} DA\,D\bar{C}\,DC\,e^{i(S[A,C,\bar{C}]+AJ+\bar{\eta}C+\bar{C}\eta)}\,. \tag{2.4}$$

By the notation $\int_{a,c,\bar{c}\,\in G}$ we indicate the integration over $A$, $C$, $\bar{C}$ with the constraint that their image under the inverse Seiberg-Witten map is in $G$, i.e. $a$, $c$, $\bar{c} \in G$. For $\mathrm{U}(N)$ the constraint is trivially solved by $A_\mu = A_\mu^M T_M$ and free integration over $A_\mu^M$, $C^M$, $\bar{C}^M$.

To explore the possibility of non-commutative $G$ Feynman rules, which after some possible projection work with the $\mathrm{U}(N)$ vertices, we write (2.4) using (2.2) as

$$Z_G[J,\bar{\eta},\eta] = e^{iS_I[\frac{\delta}{i\delta J},\frac{\delta}{i\delta\bar{\eta}},\frac{\delta}{i\delta\eta}]} Z_G^{\rm kin}[J,\bar{\eta},\eta]\,, \tag{2.5}$$

with

$$Z_G^{\rm kin}[J,\bar{\eta},\eta] = \int_{a,c,\bar{c}\in G} DA\,D\bar{C}\,DC\,e^{i(S_{\rm kin}[A,C,\bar{C}]+AJ+\bar{\eta}C+\bar{C}\eta)}\,. \tag{2.6}$$

Denoting by $\mathcal{J}$ the functional determinant for changing the integration variables from $A$, $C$, $\hat{C}$ to $a$, $c$, $\hat{c}$ we get

$$Z_G^{\rm kin}[J,\bar{\eta},\eta] = \int_{a,c,\bar{c}\in G} Da\,D\bar{c}\,Dc\,\mathcal{J}[a,c,\bar{c}]\,e^{i(S_{\rm kin}[a,c,\bar{c}]+s_1[a,c,\bar{c}]+A[a]J+\bar{\eta}C[c,a]+\bar{c}\eta)}\,. \tag{2.7}$$

The new quantity $s_1[a,c,\bar{c}]$ appearing above is defined via (1.4), (2.1) and (2.3) by

$$S_{\rm kin}[A[a],C[c,a],\bar{c}] = S_{\rm kin}[a,c,\bar{c}] + s_1[a,c,\bar{c}]\,. \tag{2.8}$$

The logarithm of (2.7) divided by $Z_G^{\rm kin}[0,0,0]$ is the generating functional for the connected Green functions of the composites $A, C, \bar{C}$ in the field theory with elementary fields $a, c, \bar{c}$ interacting via $s_1 - i\log\mathcal{J}$. Therefore it can be represented by

$$\log\left(\frac{Z_G^{\rm kin}[J,\bar{\eta},\eta]}{Z_G^{\rm kin}[0]}\right) = \sum_n i^n \int dx_1\ldots dx_n\ J(x_1)\ldots J(x_n)\langle A(x_1)\ldots A(x_n)\rangle_{\rm kin} + \cdots\,, \tag{2.9}$$

where $\langle A(x_1)\ldots A(x_n)\rangle_{\rm kin}$ stands for the $n$-point connected Green function of $A$ in this field theory. The dots at the end represent the corresponding ghost and mixed ghost and gauge field terms.

[3]To keep the notation compact we make no distinction between the group and its Lie algebra and understand space-time integration and internal index summation in the source terms.

Neglecting $\mathcal{J}$ (for a justification see next section) these are just the Green functions for the composites $A[a], C, \bar{C}$ obtained within $\theta$-expanded perturbation theory by partial summation of all diagrams built with vertices generated by the $\theta$-expansion of the non-commutative kinetic term only.

In the $\mathrm{U}(N)$-case $Z^{\text{kin}}_{\mathrm{U}(N)}[J,\bar{\eta},\eta]$ as given by (2.6) is a trivial gaussian integral. It is the generating functional of Green functions for $A$, $C$, $\bar{C}$ treated as free fields. Then in (2.9) only the two point functions $\langle AA\rangle_{\text{kin}}$ and $\langle C\bar{C}\rangle_{\text{kin}}$ are different from zero. In addition they are equal to the free propagators. Inserting this into (2.5) yields the non-commutative Feynman rules for $\mathrm{U}(N)$.

Starting with free fields and imposing a constraint, in the generic case, generates an interacting theory. We want to decide what happens in our case (2.6) for $G \subset \mathrm{U}(N)$. By some special circumstance it could be that only the connected two-point functions are modified. Another less restrictive possibility would be that connected $n$-point functions beyond some finite $n_0 > 2$ vanish. In both cases from (2.5) we would get Feynman rules with a finite number of building blocks.

For $\mathrm{U}(N)$ the equivalent representation (2.7) is due to a simple field redefinition of a free theory. Therefore, looking at the $n$-point functions of the, in terms of $a, c, \bar{c}$, composite operators $A, C, \bar{C}$ (see (1.4), (2.1)) the summation of the perturbation theory with respect to $s_1[a,c,\bar{c}] - i\log\mathcal{J}$ must yield the free field result guaranteed by (2.6).[4]

On the other side for $G \subset \mathrm{U}(N)$ we cannot directly evaluate (2.6) and are forced to work with (2.7). It will turn out to be useful to study both $\mathrm{U}(N)$ and $G \subset \mathrm{U}(N)$ in parallel. Since the result for $\mathrm{U}(N)$ is a priori known, one has some checks for the calculations within the $s_1$-perturbation theory.

## 3. $s_1$-perturbation theory for $\mathrm{U}(N)$ and $G \subset \mathrm{U}(N)$

In both cases our gauge fields take values in the Lie algebra of $\mathrm{U}(N)$. We write

$$A_\mu = A_\mu^B\, T_B \tag{3.1}$$

and use the following relations for the generators $T_A$ of the $\mathrm{U}(N)$ Lie algebra

$$[T_A, T_B] = if_{ABC}\; T_C\,, \qquad \{T_A, T_B\} = d_{ABC}\; T_C\,, \qquad \mathrm{tr}(T_A T_B) = \frac{1}{2}\delta_{AB}\,. \tag{3.2}$$

Then (1.4) and (2.1) imply

$$\begin{aligned} A_\mu^M &= a_\mu^M - \frac{1}{2}\theta^{\alpha\beta} a_\alpha^P \partial_\beta a_\mu^Q\; d_{MPQ} + \frac{1}{4}\theta^{\alpha\beta} a_\alpha^P \partial_\mu a_\beta^Q\; d_{MPQ} - \\ &\quad - \frac{1}{4}\theta^{\alpha\beta} a_\alpha^P a_\beta^Q a_\mu^R\; d_{MPS} f_{SQR} + O(\theta^2) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} C^M &= c^M + \frac{1}{4}\theta^{\alpha\beta}\partial_\alpha c^P a_\beta^Q\; d_{MPQ} + O(\theta^2) \\ \bar{C}^M &= \bar{c}^M\,. \end{aligned} \tag{3.4}$$

[4]This is a manifestation of the equivalence theorem [19].

In the case $G \subset \mathrm{U}(N)$, $G \neq \mathrm{U}(M)$, $M < N$[5] we indicate the *generators spanning the Lie algebra of* $G$ with a *lower case* Latin index and the *remaining* ones with a *primed lower case* Latin index. Upper case Latin indices run over all $\mathrm{U}(N)$ generators. Since $G$ is a subgroup and since $\{T_a, T_b\}$ is not in the Lie algebra of $G$ we have

$$f_{abc'} = 0\,, \forall\, a, b, c' \qquad \text{and} \qquad d_{abc'} \neq 0 \qquad \text{for some } a, b, c'\,. \tag{3.5}$$

As discussed in the previous section the non-commutative $G$ gauge field theory is then defined by unconstrained functional integration over $a^b_\mu, c^b, \bar{c}^b$ *and*

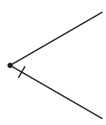
$$a^{b'}_\mu = c^{b'} = \bar{c}^{b'} = 0\,. \tag{3.6}$$

In spite of (3.6) via (3.3)–(3.5) one has non-vanishing $A^{b'}_\mu$ and $C^{b'}$.

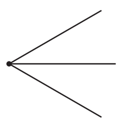
We are interested in (2.7), i.e. the Green functions of $A$, $C$, $\bar{C}$, which are composites in terms of $a$, $c$, $\bar{c}$. For the diagrammatic evaluation one gets from (3.3) the *external* vertices where all momenta are directed to the interaction point and a slash denotes derivative of the field at the corresponding leg. (We write down the $\propto \theta^0$ and $\propto \theta^1$ contributions only. Momentum conservation at all vertices is understood.)

$p, \mu, M$  $k, \alpha, A$

$$= \begin{cases} \delta^\alpha_\mu\ \delta_{AM} & \text{for } M = m \\ 0 & \text{for } M = m' \end{cases}\,, \tag{3.7}$$

$p, \mu, M$  $k_2, \beta, B$; $k_1, \alpha, A$

$$= i\left(\frac{1}{4}\theta^{\beta\alpha}\delta^\nu_\mu d_{MAB} - \frac{1}{2}\theta^{\beta\nu}\delta^\alpha_\mu d_{MAB}\right)(k_1)_\nu\,, \tag{3.8}$$

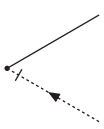
$p, \mu, M$  $k_3, \gamma, C$; $k_2, \beta, B$; $k_1, \alpha, A$

$$= -\frac{1}{4}\,\theta^{\alpha\beta} d_{MAE} f_{EBC}\delta^\gamma_\mu\,, \tag{3.9}$$

and

$p, M$  $k, \alpha, A$ $=$ $p, M$  $k, A$

$$= \begin{cases} \delta_{AM} & \text{for } M = m \\ 0 & \text{for } M = m' \end{cases}\,, \tag{3.10}$$

$p, M$  $k_2, \beta, B$; $k_1, A$

$$= \frac{i}{4}\theta^{\nu\beta} d_{MAB}(k_1)_\nu\,. \tag{3.11}$$

[5]In the following we sometimes implicitly understand that $G \subset \mathrm{U}(N)$ excludes $\mathrm{U}(M)$ subgroups.

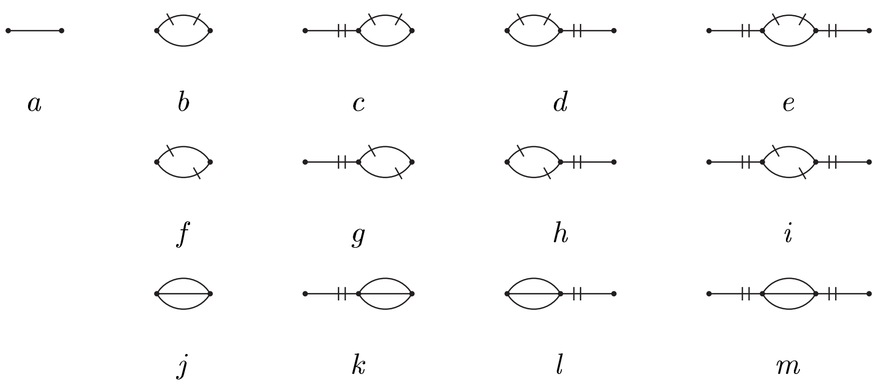
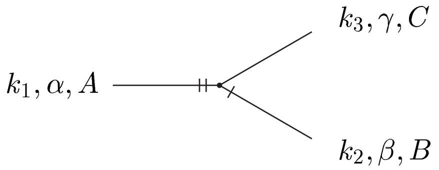


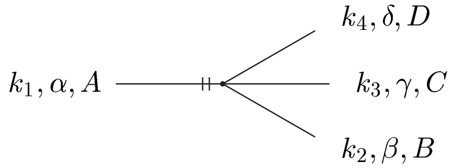

**Figure 1:** Contributions to $\langle A^M_\mu A^N_\nu \rangle_{\rm kin}$ up to order $\theta^2$


**Figure 2:** Some additional vanishing contributions to $\langle A^M_\mu A^N_\nu \rangle_{\rm kin}$

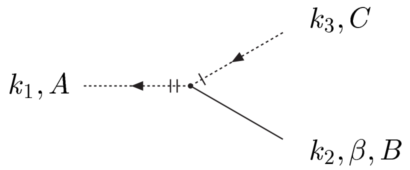
The insertion of (3.3) and (3.4) into (2.8) yields $s_1[a, c, \bar{c}]$ generating the *internal* vertices




$$= \frac{1}{g^2}\left(\frac{1}{4}\theta^{\gamma\beta}g^{\nu\alpha} - \frac{1}{2}\theta^{\gamma\nu}g^{\alpha\beta}\right) d_{ABC}k_1^2(k_2)_\nu \,, \tag{3.12}$$




$$= \frac{i}{4g^2}\theta^{\delta\gamma}d_{ADE}f_{ECB}g^{\alpha\beta}k_1^2 \,, \tag{3.13}$$




$$= -\frac{1}{4}\theta^{\beta\nu}d_{ABC}(k_3)_\nu k_1^2 \,. \tag{3.14}$$

The double slash stems from the derivatives in (2.3) after partial integration and denotes the action of $\Box = \partial_\mu \partial^\mu$ at the corresponding leg.

The propagators are

$$-ig^2 g_{\alpha\beta}\delta_{AB}\frac{1}{k^2}\,, \qquad i\delta_{AB}\frac{1}{k^2} \tag{3.15}$$

for the commuting gauge field and ghosts, respectively.

Up to now we have not taken into account the functional determinant $\mathcal{J}$ in (2.7). To simplify the analysis we use dimensional regularization. Then this determinant is equal to one, and all diagrams containing momentum integrals not depending on any external

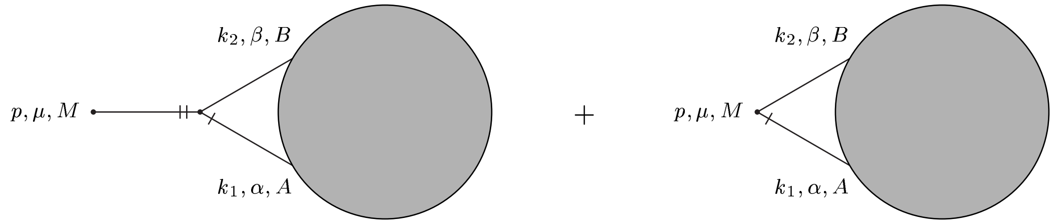



**Figure 3:** Cancelling graphs from the $\propto a^2$ terms of the $\propto \theta$ terms of the SW map

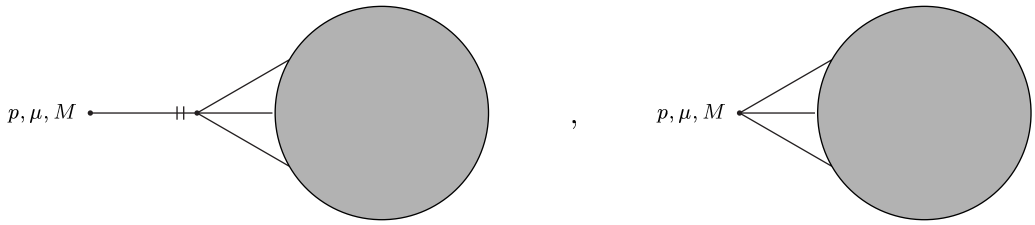



**Figure 4:** Cancelling graphs from the $\propto a^3$ terms of the $\propto \theta$ terms of the SW map

momentum or mass parameter (tadpole type) are zero, see e.g. [20]. For other regularizations these tadpole type diagrams just cancel the determinant contributions, at least in the U($N$) case.

After these preparations we consider the 2-point function $\langle A^M_\mu A^N_\nu \rangle_{\rm kin}$ within the perturbation theory with respect to $s_1$, see (2.7). Figure 1 shows all diagrams up to order $\theta^2$ which do not vanish in dimensional regularization. To give an impression, figure 2 presents a part of the remaining vanishing diagrams.

Let us first continue with the U($N$) case. Then a straightforward analysis shows that the diagrams in figure 1$b$–1$e$ cancel among each other. The same is true for figure 1$f$–1$i$ and for figure 1$j$–1$m$.

The cancellation mechanism is quite general. Let us denote by $M(k_1, \alpha, A|k_2, \beta, B)$ an arbitrary sub-diagram with two marked legs, denoted by a shaded bubble in figure 3. Then the sum of the two diagrams in figure 3 is equal to

$$-\frac{ig^2}{p^2} g_{\mu\nu}\delta_{MN}\frac{1}{g^2}\left(\frac{1}{4}\theta^{\beta\alpha}g^{\lambda\nu} - \frac{1}{2}\theta^{\beta\lambda}g^{\nu\alpha}\right) d_{NAB}p^2(k_1)_\lambda M(k_1, \alpha, A|k_2, \beta, B) + \\ + i\left(\frac{1}{4}\theta^{\beta\alpha}\delta^\lambda_\mu - \frac{1}{2}\theta^{\beta\lambda}\delta^\alpha_\mu\right) d_{MAB}(k_1)_\lambda M(k_1, \alpha, A|k_2, \beta, B) = 0\,. \quad (3.16)$$

A similar general cancellation mechanism holds for diagrams of the type shown in figure 4.

Altogether relying only on the $s_1$-perturbation theory we have convinced ourselves that for U($N$)

$$\langle A^M_\mu A^N_\nu \rangle_{\rm kin} = -ig^2 g_{\mu\nu}\delta_{AB}\frac{1}{p^2} + O(\theta^3)\,. \quad (3.17)$$

Of course, from the representation (2.6) we know a priori that there are in all orders of $\theta$ no corrections to the free propagator. Nevertheless the above exercise was useful, since it unmasked the cancellation mechanism for figure 3 and figure 4 as being essential for establishing the already known result purely within $s_1$-perturbation theory. It is straightforward to check also the vanishing of connected $n$-point functions for $n > 2$.

What changes if we switch from $\mathrm{U}(N)$ to $G \subset \mathrm{U}(N)$? First of all, then we do not know the answer in advance and have to rely only on $s_1$-perturbation theory. Secondly, in this perturbation theory the above cancellation mechanism is set out of work for external points carrying a primed index, related to Lie algebra elements of $\mathrm{U}(N)$ not in the Lie algebra of $G$. Then according to (3.7) the external vertex to start with in the first diagrams of figure 3 and figure 4 is zero, i.e. the partner to cancel the second diagrams disappears. This observation is a strong hint that for $G \subset \mathrm{U}(N)$ there remain non-vanishing connected Green functions $\langle A(x_1)\dots A(x_n)\rangle_{\mathrm{kin}}$ for all integer $n$. An explicit proof will be given in the next section.

## 4. Non-vanishing $n$-point Green functions generated by $\log Z_G^{\mathrm{kin}}$

The connected Green functions

$$G_n^{\mathrm{kin},M_1\dots M_n}(x_1,\dots,x_n) = \Big\langle A^{M_1}(a(x_1))\dots A^{M_n}(a(x_n))\Big\rangle_{\mathrm{kin}}$$

are power series in $\theta$ and $g$. To prove their non-vanishing for generic $\theta$ and $g$ it is sufficient to extract at least one non-zero contribution to $G_n^{\mathrm{kin}}$ of some fixed order in $\theta$ and $g$.

To find for our purpose the simplest tractable component of the Green function it turns out to be advantageous to restrict all of the group indices $M_i$ to primed indices that do not correspond to generators of the Lie algebra of $G$. Then the Green function simplifies in first nontrivial order of the Seiberg-Witten map to:

$$\Big\langle \prod_{i=1}^{n}\Big[A^{(2)m_i'}(a(x_i)) + A^{(3)m_i'}(a(x_i))\Big]\Big\rangle_{\mathrm{kin}}. \tag{4.1}$$

Here $A^{(2)}$, $[A^{(3)}]$ denote the $\propto\theta$ part of the Seiberg-Witten map (3.3) with quadratic, [cubic] dependence on the ordinary field $a$. Thus the above function is $\mathcal{O}\big(\theta^n\big)$. Focussing now on the special contribution which is exactly $\propto\theta^n$, it is clear that in addition to the external vertices further $\theta$-dependence (e.g. higher order corrections to the Seiberg-Witten map) is not allowed. That means this special part of the connected Green function is universal with respect to the $\theta$-expansion of the constraint (1.4) where $a\in G$.

The special contribution to the Green function $\propto\theta^n$ then consists of $n$ to $\frac{3}{2}n$, $[\frac{3}{2}(n-1)+1]$ internal lines for $n$ even, [odd]. Two or three of these originate from each of the $n$ points (external vertices). There are no further internal vertices present stemming from the interaction term $s_1[a,c,\bar{c}]$ in (2.7) since this would increase the power in $\theta$.

In our normalization where the coupling constant $g$ is absorbed into the fields each propagator enlarges the power of the diagram in $g$ by $g^2$. Thus for general coupling $g$ it is sufficient to check the non-vanishing of all connected diagrams with the same number of propagators. Here we choose the minimum case of $n$ propagators where we can neglect all contributions from $A^{(3)}$ in (4.1). Then it follows that the connected $\propto\theta^n g^{2n}$ contributions to the Green function are given by the type of diagrams shown in figure 5.

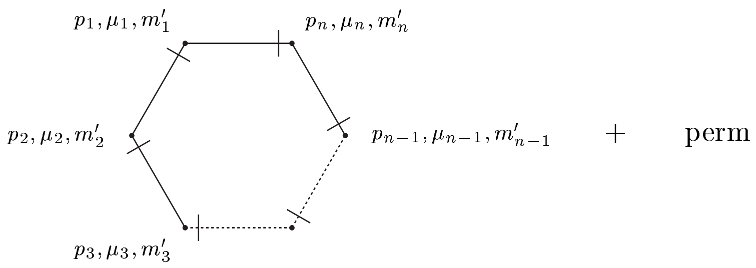



**Figure 5:** Graphs $\propto \theta^n g^{2n}$ of the connected $n$-point Green function

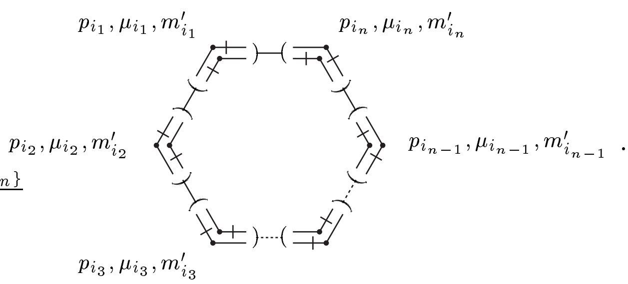
The total number of the diagrams can be determined as follows: The two lines starting at each point are distinguishable due to the derivative at one leg. To construct all connected contributions we connect the first leg of the first external vertex to one of the $2n-2$ other legs that do not start at the same external point. The next one is connected to one of the remaining $2n-4$ allowed legs, such that no disconnected subdiagram is produced and so on. We thus have to add-up $(2n-2)!! = (n-1)!\ 2^{n-1}$ diagrams. All of them can be drawn like the one shown in figure 5 by permuting the external momenta, Lorentz and group indices and the internal legs.

To sum-up all diagrams it is convenient to define two classes of permutations: The first includes all permutations that interchange the two distinct legs at one or more external vertices with the distribution of the external momenta, Lorentz and group indices held fixed. The second contains all permutations which interchange the external quantities such that this cannot be traced back to a permutation of the distinct lines at the external vertices. We call its elements proper permutations in the following.

In total $2^n$ combinations exist, generated by interchanging the distinct legs when the external points are fixed. The proper permutations are the ones which are not identical under (anti)cyclic permutations. There are $n!$ configurations of the external points and with each one $n-1$, $[n]$ others are identified under cyclic, [anticyclic] permutations, i.e. there are $\frac{n!}{2n} = \frac{(n-1)!}{2}$ proper permutations. This is consistent with the total number of diagrams.

The connected $\propto \theta^n g^{2n}$ contributions to the momentum space Green function can thus be cast into the following form:

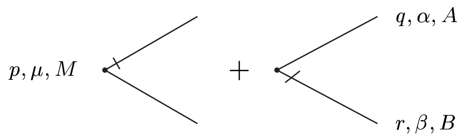
$$G_{n\ \mu_1\ldots\mu_n}^{\mathrm{kin},m'_1\ldots m'_n}(p_1,\ldots,p_n)\big|_{\propto\theta^n g^{2n}} = \sum_{\substack{\mathrm{perm}\{i_1,\ldots,i_n\} \\ \text{(anti)cycl.}}}$$




(4.2)

Here the brackets around the external vertices denote a sum over both configurations where the two legs are interchanged. These $n$ sums are then multiplied, describing exactly the $2^n$ permutations of the distinct two legs at each vertex.

The sum of the two permutations at one external vertex occurring $n$ times in (4.2) reads




$$= -\frac{i}{4}\, d_{abm'} \left[2\left(\theta^{\beta\gamma} q_\gamma \delta^\alpha_\mu + \theta^{\alpha\gamma} r_\gamma \delta^\beta_\mu\right) + \theta^{\alpha\beta}(q_\mu - r_\mu)\right].$$

Using this, the analytic expression for the $\propto \theta^n g^{2n}$ part of the connected Green function is given by

$$G_{n\ \mu_1\ldots\mu_n}^{\mathrm{kin},m'_1\ldots m'_n}(p_1,\ldots,p_n)\big|_{\propto\theta^n g^{2n}} =$$
$$= \sum_{\substack{\mathrm{perm}\{i_1,\ldots,i_n\} \\ \mathrm{(anti)cycl.}}} \frac{g^{2n}}{4^n} \int \frac{d^D k}{(2\pi)^D} \prod_{r=1}^{n} d_{a_r a_{r+1} m'_{i_r}} \Big[ -2\theta^{\alpha_r\gamma_r}(q_{r-1})_{\gamma_r} g_{\mu_{i_r}\alpha_{r+1}} + 2\theta_{\alpha_{r+1}}{}^{\gamma_r}(q_r)_{\gamma_r}\delta^{\alpha_r}_{\mu_{i_r}} -$$
$$- \theta^{\alpha_r}{}_{\alpha_{r+1}}(q_{r-1}+q_r)_{\mu_{i_r}}\Big] \frac{1}{q^2_{r-1}}\,, \tag{4.3}$$

where summation over $\alpha_r$ appearing twice in the sequence of multiplied square brackets is understood. Thereby one has to identify $a_{n+1} = a_1$, $\alpha_{n+1} = \alpha_1$, $p_{i_n} = -\sum_{r=1}^{n-1} p_{i_r}$. The $q_r$ are defined by

$$q_r = q_r(k, p_{i_1}, \ldots, p_{i_r}) = k + \sum_{s=1}^{r} p_{i_s}\,. \tag{4.4}$$

In appendix A we prove that this expression is indeed non-zero at least for even $n$ and the most symmetric non-trivial configuration of the external momenta, Lorentz and group indices. This means that non-vanishing connected $n$ point functions for arbitrary high $n$ exist in the kinetic perturbation theory, leading to infinitely many building blocks in the $\theta$-summed case. In other words one needs infinitely many elements to formulate Feynman rules for the non-commutative $G$-gauge theory if one insists on keeping the non-commutative $\mathrm{U}(N)$ vertices as components.

Due to the fact that the expressions discussed above cannot be affected by higher order corrections of (1.4) this statement is universal, i.e. independent of the power in $\theta$ to which the constraint $a \in G$ is implemented.

## 5. The case with sources restricted to the Lie algebra of $G$

Up to now we have looked for Feynman rules working with the original $\mathrm{U}(N)$ vertices and sources $J^M$ taking values in the full $\mathrm{U}(N)$ Lie algebra. This seemed to be natural since in the enveloping algebra approach for $G \subset \mathrm{U}(N)$ the non-commutative gauge $A^M$ field, although constrained, carries indices $M$ running over all generators of $\mathrm{U}(N)$.

There is still another option to explore. First one can restrict the sources $J$, $\eta$, $\bar\eta$ in (2.4) by hand to take values in the Lie algebra of $G$ only. Then instead of pulling out in (2.5) the complete interaction $S_I$ one separates only those parts of $S_I$, which yield vertices whose external legs carry lower case Latin indices referring to the Lie algebra of $G$ exclusively. The remaining parts of $S_I$, generating vertices with at least one leg owning a primed index, are kept under the functional integral. The functional integration and the constraint remain unchanged. We denote this splitting of $S_I$ by

$$S_I[A,C,\bar C] = S_i[A,C,\bar C] + S_i'[A,C,\bar C] \tag{5.1}$$

and the sources by hatted quantities

$$\widehat{J}^{a'} = \widehat{\bar\eta}^{a'} = \widehat{\eta}^{a'} = 0\,. \tag{5.2}$$

Then

$$Z_G[\widehat{J},\widehat{\bar\eta},\widehat{\eta}] = e^{iS_i[\frac{\delta}{i\delta\widehat{J}},\frac{\delta}{i\delta\widehat{\bar\eta}},\frac{\delta}{i\delta\widehat{\eta}}]}\;\widehat{Z}_G[\widehat{J},\widehat{\bar\eta},\widehat{\eta}] \tag{5.3}$$

and

$$\begin{aligned}\widehat{Z}_G[\widehat{J},\widehat{\bar\eta},\widehat{\eta}] &= \int_{a,c,\bar c\in G} DA\, D\bar C\, DC\, e^{i(S_{\rm kin}[A,C,\bar C]+S_i'[A,C,\bar C]+A\widehat{J}+\widehat{\bar\eta}C+\bar C\widehat{\eta})}\\ &= \int_{a,c,\bar c\in G} Da\, D\bar c\, Dc\, \mathcal{J}\, e^{i(S_{\rm kin}[a,c,\bar c]+\widehat{s}_1[a,c,\bar c]+A[a]\widehat{J}+\widehat{\bar\eta}C[c,a]+\bar c\widehat{\eta})}\,,\end{aligned} \tag{5.4}$$

where $\widehat{s}_1[a,c,\bar c]$ is defined by

$$S_{\rm kin}[A[a],C[c,a],\bar c] + S_i'[A[a],C[c,a],\bar c] = S_{\rm kin}[a,c,\bar c] + \widehat{s}_1[a,c,\bar c]\,. \tag{5.5}$$

If now

$$\begin{aligned}&\log\Big(\widehat{Z}_G[\widehat{J},\widehat{\bar\eta},\widehat{\eta}]/\widehat{Z}_G[0]\Big) =\\ &\qquad = \sum_n i^n \int dx_1\ldots dx_n\; \widehat{J}(x_1)\ldots\widehat{J}(x_n)\langle A(x_1)\ldots A(x_n)\rangle_{{\rm kin}+S_i'} + \cdots\,,\end{aligned} \tag{5.6}$$

e.g. for $G = \mathrm{SO}(N)$, in the spirit of (2.9) would generate only the free propagators, the $\mathrm{SO}(N)$ Feynman rules conjectured in ref. [17][6] would have been derived via partial summation of the $\theta$-expanded perturbation theory in the enveloping algebra approach.

In the remaining part of this section we prove that this *cannot* happen. For this purpose we consider $\langle A^{m_1}(x_1)\ldots A^{m_n}(x_n)\rangle_{{\rm kin}\,+S_i'}$ and look at it as a power series in $g^2$ and $\theta$. To prove that it is not identically zero, it is sufficient to find a particular non-vanishing order in $g^2$, $\theta$. Let us concentrate on the lowest possible order in $g^2$.

[6]There have been given arguments [7] that their constraint is equivalent to requiring the image under the inverse SW map to be in $\mathrm{SO}(N)$.

At all $x_i$ the contributing diagrams in the $\widehat{s}_1$-perturbation theory have to start with at least one commutative gauge field propagator (3.15). This generates at least a factor $g^{2n}$ (One propagator at each $x_i$ corresponding to the lowest order of SW map.) The diagrams have to be connected. To achieve this, with respect to power counting in $g^2$, in the most effective way one has to connect all the $n$ legs in just one $n$-point vertex of the $\widehat{s}_1$-perturbation theory, ending up with a total $g^2$-power of $g^{2n-2}$.

Now we search in addition for the lowest possible power in $\theta$. The $n$-point vertices arise from expressing the non-commutative fields $A$ either in the original non-commutative kinetic term or 3-point or 4-point interactions in $S_i'$ (see (5.1)) via (3.3) in terms of the commutative field $a$.[7] Let us look for the most efficient way for simultaneously trading a minimal number of $\theta$-factors combined with a maximal number of $a$-legs. Simple dimensional analysis shows that this is achieved by terms in the SW map (3.3) not containing derivatives, i.e. terms of the type $(a)^l(\theta)^{\frac{l-1}{2}}$. Then independent of the order of contribution to the SW-map $n$-point vertices originating from the kinetic term, the original 3-point or 4-point vertices behave like $\theta^{n/2-1}$, $\theta^{n/2-3/2}$ and $\theta^{n/2-2}$, respectively. From this observation we can conclude that for a given $n$ within the lowest $g^2$-power term the minimal number of $\theta$ factors is exclusively realized by connecting the $n$-external legs in just one $n$-point vertex generated by SW-mapping out of a 4-point interaction of $S_i'$.

In appendix B we prove for SO(3) that the corresponding contribution to the 8-point function $\langle A^{m_1}(x_1)\dots A^{m_8}(x_8)\rangle_{\mathrm{kin}\ +S_i'}$ is different from zero. This excludes the rules of [17].

The more ambitious program to exclude rules based on the vertices in $S_i$ and an arbitrary but finite number of additional building blocks would require to show, similar to the previous section, that there is no $n_0$ assuring vanishing connected $n$-point functions for $n > n_0$. Although we have practically no doubt concerning this conjecture, a rigorous proof is beyond our capabilities since for increasing $n$ higher and higher orders of the SW-map contribute. This happens because in contrast to the proof in section 4 one is forced to look at Green functions with all external group indices referring to generators of the Lie algebra of $G$ since no primed indices of the remaining generators spanning U($N$) are probed.

## 6. Conclusions

Starting from the enveloping algebra approach we have studied the issue of partial summation of $\theta$-expanded perturbation theory for subgroups $G \subset$ U($N$). The main motivation was given by the search for some Feynman rules exhibiting UV/IR mixing similar to the well known U($N$) case. The original Feynman rules in the enveloping algebra approach contain an infinite number of vertices. They are read off from the interactions in terms of the commutative gauge field $a_\mu$ (and ghosts) taking values in the Lie algebra of $G$. Our aim was to decide, whether by some partial summation new rules related to the interactions of the non-commutative gauge field $A_\mu$ (and ghosts) can be derived. The non-commutative fields take values in the Lie algebra of U($N$), but are constrained to be related to the commuta-

[7]Note that the original 3-point or 4-point interactions by themselves are $\theta$-dependent via the $\star$-product. But since we are searching for lowest order in $\theta$ this further $\theta$-dependence can be disregarded.

tive fields by the Seiberg-Witten map. Coming from the side of $\theta$-expanded perturbation theory the non-commutative fields are composites constructed out of the commutative fields.

With our initial formula (2.5) we have decided to choose the vertices generated by the interaction term $S_I$ in terms of $A_\mu, C, \bar{C}$ as part of the building blocks of the wanted Feynman rules. Then the remaining ingredients are given by the connected Green functions related to $Z_G^{\mathrm{kin}}$ in (2.6). We found that for $G \subset \mathrm{U}(N)$, $G \neq \mathrm{U}(N)$, $M < N$, the number of legs of non-vanishing connected Green functions generated by $\log Z_G^{\mathrm{kin}}$ is not bounded from above. Therefore, there are no Feynman rules based on the $A, C, \bar{C}$ vertices in $S_I$ and, besides perhaps suitable modified propagators, at most a finite number of additional building blocks with gauge field or ghost legs.

As usual in the case of no go theorems one has to be very carefully in stressing the input made. Our negative statement is bound to the a priori decision to work with the $A, C, \bar{C}$ vertices from $S_I$. Of course we cannot exclude at this stage the existence of rules exhibiting UV/IR mixing based on some clever modification of these vertices. We also cannot exclude that the infinite set of building blocks with gauge field and ghost legs by means of some additional auxiliary field could be resolved into rules with only a finite number of building blocks.

In the SO($N$) case Feynman rules for the fields $A$, $C$, $\bar{C}$ have been conjectured in ref. [17]. In these rules vertices and propagators carry only SO($N$) indices. To make contact with this situation we have modified the set up of eqs. (2.5), (2.6) to (5.3), (5.4). This ensures that one already has the vertices of [17] as building blocks. A derivation of these rules would then require that $\log \widehat{Z}_G$ would generate nothing beyond a connected two point function. However, at least for SO(3) we were able to show explicitly that there is a non-vanishing connected 8-point function.

## Acknowledgments

We thank R. Helling, M. Salizzoni and P. Schupp for useful discussions. The work of C. S. is supported by DFG via the Graduiertenkolleg 271, that of H. D. in part by DFG via the project DO 447/3-1 and Graduiertenkolleg 271.

## A. Proof of the non-vanishing of (4.3)

To prove that the Green function in (4.3) is non-zero, it is sufficient to show that at least one contribution to this quantity with an *independent* tensor structure is non-vanishing at some configuration of the external momenta, Lorentz and group indices. Choosing the most symmetric non-trivial external configuration

$$p_1 = \cdots = p_{n-1} = p\,, \qquad p_n = -(n-1)p\,, \qquad \mu_1 = \cdots = \mu_n = \mu\,, \qquad m'_1 = \cdots m'_n = m' \tag{A.1}$$

simplifies (4.3) considerably, e.g. the summation over permutations of the external quantities simply leads to a combinatorial factor.

We first pick out all terms where — *after* performing the integral of (4.3) — the tensor structure of the $\mu_i$ is purely constructed with $g_{\mu_i\mu_j}$ such that $\theta^{\alpha\beta}$ does not carry an external Lorentz index $\mu_i$. To minimize the number of contributing terms we choose $\theta^{\alpha\beta}(p_i)_\beta = 0$.[8] In this case the square brackets in (4.3) simplify and we use the abbreviations

$$2\Big[\textcircled{1}_r + \textcircled{2}_r + \textcircled{3}_r\Big]k = 2\Big[-\theta^{\alpha_r\gamma_r} g_{\mu_{i_r}\alpha_{r+1}} + \theta_{\alpha_{r+1}}{}^{\gamma_r}\delta^{\alpha_r}_{\mu_{i_r}} - \theta^{\alpha_r}{}_{\alpha_{r+1}}\delta^{\gamma_r}_{\mu_{i_r}}\Big]k_{\gamma_r}\,,$$

where Lorentz indices are not written explicitly. For the three terms inside the bracket only the following multiplications can produce a pure $g_{\mu_i\mu_j}$-structure

$$\begin{aligned}
\textcircled{1}_r\textcircled{2}_{r+1} &= \theta^{\alpha_r\gamma_r}\theta^{\gamma_{r+1}}{}_{\alpha_{r+2}} g_{\mu_{i_r}\mu_{i_{r+1}}}\\
\textcircled{2}_r\textcircled{1}_{r+1} &= {}^{\gamma_r}\theta\theta^{\gamma_{r+1}}\delta^{\alpha_r}_{\mu_{i_r}} g_{\mu_{i_{r+1}}\alpha_{r+2}}\\
\textcircled{2}_r\textcircled{3}_{r+1} &= {}^{\gamma_r}\theta\theta_{\alpha_{r+2}}\delta^{\alpha_r}_{\mu_{i_r}}\delta^{\gamma_{r+1}}_{\mu_{i_{r+1}}}\\
\textcircled{3}_r\textcircled{1}_{r+1} &= {}^{\alpha_r}\theta\theta^{\gamma_{r+1}}\delta^{\gamma_r}_{\mu_{i_r}} g_{\mu_{i_{r+1}}\alpha_{r+2}}\\
\textcircled{3}_r\textcircled{3}_{r+1} &= {}^{\alpha_r}\theta\theta_{\alpha_{r+2}}\delta^{\gamma_r}_{\mu_{i_r}}\delta^{\gamma_{r+1}}_{\mu_{i_{r+1}}}\,,
\end{aligned}$$

where we have defined ${}^\alpha\theta\theta^\gamma = \theta^{\alpha\beta}\theta_\beta{}^\gamma$. These products are the building blocks of the complete terms with $n$ factors, for instance like

$$\underbrace{\textcircled{1}_1\textcircled{2}_2\cdots\textcircled{1}_{k-1}\textcircled{2}_k\textcircled{3}_{k+1}\cdots\textcircled{3}_{j+k}\cdots}_{n}\,,$$

where the $\alpha_1$ index of the first factor is contracted with the $\alpha_{n+1}$ index of the last.

Further restrictions are imposed on the complete expressions: The total number of factors $n$ has to be even because one cannot construct a pure $g_{\mu_i\mu_j}$ structure with an odd number of $\mu_i$'s. In addition the number of ③'s in the complete product of $n$ terms has to be even as otherwise after performing the integral in (4.3) one $\theta$ would carry an index $\mu_i$ (see equations below). Then it follows that the numbers of ①'s and ②'s have to be identical.

Using the configuration (A.1) the contribution of all terms with an even number $j$ of ③'s and an even number $n-j$ ①'s and ②'s can now be written as

$$\begin{aligned}
&G_n^{\mathrm{kin},m'\dots m'}{}_{\mu\dots\mu}(p,\dots,p,-(n-1)p)\big|_{\propto\theta^n g^{2n},\ \text{only } g_{\mu\mu}} = \\
&= \frac{(n-1)!}{2}\frac{g^{2n}}{2^n}\left[\prod_{r=1}^{n} d_{a_r a_{r+1} m'}\right]\times \\
&\quad\times\sum_{j=0,2}^{n}\underbrace{(\theta\dots\theta)}_{n}{}^{\gamma_{j+1}\cdots\gamma_n}\underbrace{g_{\mu\mu}\cdots g_{\mu\mu}}_{n-j}\delta^{\gamma_1}_{\mu}\dots\delta^{\gamma_j}_{\mu}\int\frac{d^Dk}{(2\pi)^D}\frac{k_{\gamma_1}\dots k_{\gamma_n}}{q_1^2\dots q_n^2}\Big|_{\text{only special } g}\,,
\end{aligned}\tag{A.2}$$

where the factor $\frac{(n-1)!}{2}$ stems from performing the summation over all proper permutations and $q_r = k + rp$, $r \neq n$, $q_n = k$. To make the above expression compact we have used some further abbreviations which we now explain.

[8]This can be realized for the choice (A.1).



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The relevant part of the integral in the above expression is defined as the tensor component of the integral only made out of the metric where the metric must not possess a mixed index pair with one index from the set $\{\gamma_1, \ldots, \gamma_j\}$ and one from the set $\{\gamma_{j+1} \ldots \gamma_n\}$. It then reads

$$\int \frac{d^D k}{(2\pi)^D} \frac{k_{\gamma_1} \ldots k_{\gamma_n}}{q_1^2 \ldots q_n^2}\bigg|_{\text{only special } g} = I_0 \sum_{\substack{\text{perm} \\ \{i_1,\ldots,i_j\} \\ \{i_{j+1},\ldots,i_n\}}} \prod_{r=1,3}^{n-1} g_{\gamma_{i_r}\gamma_{i_{r+1}}} , \tag{A.3}$$

where $I_0$ denotes a scalar integral which will be discussed later.

The tensor $(\theta \ldots \theta)^{\gamma_{j+1}\cdots\gamma_n}$ in (A.2) is build by summing over all possibilities to replace $\frac{n-j}{2}$ of the $n$ summation index pairs $(\alpha_r, \alpha_r)$ in the trace $\operatorname{tr}\{\theta^n\} = \theta^{\alpha_1}{}_{\alpha_2}\theta^{\alpha_2}{}_{\alpha_3} \ldots \theta^{\alpha_n}{}_{\alpha_1}$ by the index pairs $\{(\gamma_{j+1}, \gamma_{j+2}), \ldots, (\gamma_{n-1}, \gamma_n)\}$ keeping the ordering of the $\gamma$-pairs, i.e. the pair $(\gamma_{j+1}, \gamma_{j+2})$ is inserted at the positions with smallest index $r$ of all replaced $\alpha_r$ and so on. All indices $r$ of the replaced $\alpha_r$ either have to be odd or even, since otherwise at least two substructures ${}^{\gamma_r}\theta \ldots \theta^{\gamma_{r+1}}$ would contain an odd number of $\theta$'s vanishing when contracted with the symmetric $k_{\gamma_r}k_{\gamma_{r+1}}$ in (A.2). Some examples for illustration: If $j = n$ in (A.2) then $(\theta \ldots \theta) = \operatorname{tr}\{\theta^n\}$ and there is only one contribution. If $j = n-2$ then there are $n$ possibilities[9] to replace a pair $\alpha_r$ by the pair $(\gamma_{n-1}, \gamma_n)$ such that $(\theta \ldots \theta)^{\gamma_{n-1}\gamma_n} = n\ {}^{\gamma_{n-1}}\theta \ldots \theta^{\gamma_n}$. For general $j \neq n$ there are $2\binom{n/2}{j/2}$ non-vanishing possibilities to replace summation indices by the $\gamma$-pairs.

The contraction of the above defined $(\theta \ldots \theta)^{\gamma_{j+1}\cdots\gamma_n}$ in (A.2) with the tensor structure of the integral (A.3) leads to a sum over products of traces of the form $\prod_i \operatorname{tr}\{\theta^{2k_i}\}$, $k_i \in \mathbb{N}$ such that $\sum_i 2k_i = n$. All these products of traces include the same sign $(\operatorname{sgn} \operatorname{tr}\{\theta^2\})^{\frac{n}{2}}$.[10] Thus all summed terms in (A.2) carry the same sign such that a cancellation mechanism between different terms cannot be present. Proving the non-vanishing of (A.2) therefore only requires to show that the group structure factor and the scalar integral $I_0$ in (A.3) are non-zero.

For instance, the choice $m' = N^2$, where the generator $T_{N^2}$ is given by $T_{N^2} = \frac{1}{\sqrt{2N}}\mathbf{1}$ in an $\mathrm{U}(N)$ theory, leads to $d_{abN^2} = \sqrt{\frac{2}{N}}\delta_{ab}$. Hence, with $\dim G$ as the dimension of the Lie algebra of $G$

$$\prod_{r=1}^{n} d_{a_r a_{r+1} m'}\bigg|_{m'=N^2} = \left(\frac{2}{N}\right)^{\frac{n}{2}} \dim G$$

does not vanish.

In general the integral in (A.2) can be decomposed in scalar integrals like

$$\int \frac{d^D k}{(2\pi)^D} \frac{k_{\gamma_1} \ldots k_{\gamma_n}}{q_1^2 \ldots q_n^2} = I_0 \sum_{\substack{\text{perm} \\ \{i_1,\ldots,i_n\}}} \prod_{r=1,3}^{n-1} g_{\gamma_{i_r}\gamma_{i_{r+1}}} + \text{ terms containing } p_{\gamma_i}$$

where due to the choice (A.1) the $q_i$ (4.4) now only depend on $p$ such that the above tensor structure can only be spanned by $g_{\gamma_i\gamma_j}$ and $p_{\gamma_i}$. Notice that in (A.2) only one part of the

[9] $\frac{n}{2}$ possibilities to replace $\alpha_r$ with odd $r$ and $\frac{n}{2}$ to replace the ones with even $r$.

[10] This can be proven by using the canonical skew-diagonal form of [2].

total symmetric tensor multiplying $I_0$ given in (A.3) is needed. In the above expression we now choose all indices $\gamma_1 = \dots = \gamma_n = \gamma$ and the momentum $p$ such that it has a vanishing component $p_\gamma$ for the special choice of $\gamma$. Then one finds for $I_0$

$$I_0 = \frac{1}{n!} \int \frac{d^D k}{(2\pi)^D} \frac{(k_\gamma)^n}{q_1^2 \dots q_n^2} ,$$

For even $n$ this is non-vanishing since it is positive definite after a Wick rotation.

Thus the expression (A.2) in general does not vanish for all even $n$ implying that at least all connected $n$-point Green functions with an even number of external points are therefore present in the kinetic theory such that it produces infinitely many building blocks in the $\theta$-summed case.

## B. Proof of the non-vanishing of $\langle A^{m_1}(x_1)\dots A^{m_8}(x_8)\rangle_{\mathrm{kin}\,+S_i'}$ in the SO(3) case

In this appendix we give an explicit proof for the non-vanishing of the lowest order contribution in $g^2$ and $\theta$ to $\langle A^{m_1}(x_1)\dots A^{m_8}(x_8)\rangle_{\mathrm{kin}\,+S_i'}$ in the SO(3) case.

As discussed in the main text, the contribution we are after is isolated by connecting the 8 external legs in one 8-point vertex generated out of a 4-point interaction of the non-commutative $A$ in $S_i'$. Via the definition of $S_i'$, at least one of the $A$ has to carry a primed group index. Since we look for the lowest order in $\theta$ we can replace the $\star$-product by the usual product. The interaction then has the gauge group structure $f_{N_1N_2K}f_{N_3n_4'K}$. Due to the subgroup property of $G$ this is zero if $N_j = n_j$, $j = 1,2,3$. Therefore we have to start with a 4-point interaction of the $A$ where two of them carry a primed index. Then there contribute three interaction terms

$$\begin{aligned}&\frac{i}{g^2}\left(f_{m_1m_2a}f_{n_3'n_4'a}g^{\mu_1\nu_3}g^{\mu_2\nu_4} + f_{n_4'm_1a'}f_{m_2n_3'a'}g^{\mu_1\nu_3}g^{\mu_2\nu_4} + f_{n_4'm_1a'}f_{n_3'm_2a'}g^{\mu_1\mu_2}g^{\nu_3\nu_4}\right)\times\\&\times A^{m_1}_{\mu_1}A^{m_2}_{\mu_2}A^{m_3'}_{\nu_3}A^{m_4'}_{\nu_4}\,. \qquad\qquad (\mathrm{B.1})\end{aligned}$$

Now we replace $A^{m_i}_{\mu_i}$ by $a^{m_i}_{\mu_i}$ for $i = 1,2$ and $A^{n_i'}_{\nu_i}$, $i = 3,4$ by the term with maximum number of $a$ within the $\theta^1$ contribution, see (3.3), and get

$$\begin{aligned}&\frac{i}{16g^2}\left(f_{m_1m_2a}f_{n_3'n_4'a}g^{\mu_1\mu_5}g^{\mu_2\mu_8} + f_{m_1n_4'a'}f_{m_2n_3'a'}(g^{\mu_1\mu_2}g^{\mu_5\mu_8} - g^{\mu_1\mu_5}g^{\mu_2\mu_8})\right)\times\\&\times d_{n_3'm_3e}f_{em_4m_5}d_{n_4'm_6k}f_{km_7m_8}\theta^{\mu_3\mu_4}\theta^{\mu_6\mu_7}a^{m_1}_{\mu_1}a^{m_2}_{\mu_2}\dots a^{m_8}_{\mu_8}\,. \qquad (\mathrm{B.2})\end{aligned}$$

With this interaction the $g^{2\cdot 8-2}\theta^2$ contribution to the Fourier transform of $\langle A(x_1)\dots A(x_8)\rangle_{\mathrm{kin}\,+S_i'}$ becomes up to the momentum conservation factor equal to

$$\begin{aligned}M^{\mu_1\dots\mu_8}_{m_1\dots m_8} = \frac{i}{16}g^{14}&\sum_{\substack{\text{perm}\\ \{i_1,\dots,i_8\}}}\theta^{\mu_{i_3}\mu_{i_4}}\theta^{\mu_{i_6}\mu_{i_7}}\;d_{n_3'm_{i_3}e}f_{em_{i_4}m_{i_5}}\;d_{n_4'm_{i_6}k}f_{km_{i_7}m_{i_8}}\times\\&\times\Bigg(\frac{1}{2}(g^{\mu_{i_1}\mu_{i_5}}g^{\mu_{i_2}\mu_{i_8}} - g^{\mu_{i_2}\mu_{i_5}}g^{\mu_{i_1}\mu_{i_8}})\;f_{m_{i_1}m_{i_2}a}f_{n_3'n_4'a} +\\&\quad + (g^{\mu_{i_1}\mu_{i_2}}g^{\mu_{i_5}\mu_{i_8}} - g^{\mu_{i_1}\mu_{i_5}}g^{\mu_{i_2}\mu_{i_8}})\;f_{m_{i_1}n_4'a'}f_{m_{i_2}n_3'a'}\Bigg)\,. \qquad (\mathrm{B.3})\end{aligned}$$

We will have reached the goal of this appendix if it can be shown that the above quantity is different from zero. Our explicit proof of $M^{\mu_1\ldots\mu_8}_{m_1\ldots m_8} \neq 0$ consists in the numerical calculation for one special choice of gauge group and Lorentz indices. To minimize the calculational effort forced by taking into account all the permutations, we looked for an index choice with a lot of symmetry with respect to the interchange of external legs. But we also had to avoid too much symmetry not to produce a zero result.

If we use the standard Gell-Mann enumeration of the nine generators of the U(3) Lie algebra, see e.g. [21], the generators of the SO(3) subalgebra carry the indices 2,5,7. Then our special choice for the external legs is

| | leg 1 | leg 2 | leg 3 | leg 4 | leg 5 | leg 6 | leg 7 | leg 8 |
|---|---|---|---|---|---|---|---|---|
| $m_i$: | 5 | 5 | 2 | 5 | 2 | 2 | 5 | 2 |
| $\mu_i$: | $\lambda$ | $\lambda$ | $\mu$ | $\nu$ | $\mu$ | $\mu$ | $\nu$ | $\mu$ |

The chosen Lorentz indices are all spacelike and have to fulfill

$$\begin{aligned} \mu \neq \nu\,, \qquad & \mu \neq \lambda\,, \qquad \nu \neq \lambda\,, \\ \theta^{\mu\nu} \neq 0\,, \qquad & \theta^{\mu\lambda} = 0\,, \qquad \theta^{\nu\lambda} = 0\,. \end{aligned} \tag{B.4}$$

Taking into account the list of vanishing $d_{ABC}$ and $f_{ABC}$ for U(3) [21] we find

$$M^{\mu_1\ldots\mu_8}_{m_1\ldots m_8}|_{\text{special}} = 6ig^{14}\,(\theta^{\mu\nu})^2\, f^2_{257} \left[(f_{345}d_{247} - f_{123}d_{157})^2 + f^2_{458}d^2_{247}\right]. \tag{B.5}$$

All $f$ and $d$ in (B.5) are different from zero.

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